

## Bosonization of QCD at high density

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### Abstract

We describe the color-flavor locking (CFL) color superconductor in terms of bosonic variables, where the gaped quarks are realized as solitons, so-called superqualitons. We then show that the ground state of the CFL color superconductor is a  $Q$ -matter, which is the lowest energy state for a given fixed baryon number. From this  $Q$ -matter, we calculate the minimal energy to create a superqualiton and argue that it is twice of the Cooper gap. Upon quantizing the zero modes of superqualitons, we find superqualitons have the same quantum number as the gaped quarks and furthermore all the high spin states of superqualitons are absent in the effective bosonic description of the CFL color superconductor.

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It is sometimes convenient to describe a system of interacting fermions in terms of bosonic variables, since often in that description the interaction of elementary excitations becomes weak and perturbative approaches are applicable [1]. In this paper, we attempt to bosonize cold quark matter of three light flavors, where the low-lying energy states are bosonic.

Due to asymptotic freedom [2,3], the stable state of matter at high density will be quark matter [4], which has been shown to exhibit color superconductivity at low temperature [5,6]. The color superconducting quark matter might exist in the core of neutron stars, since the Cooper-pair gap and the critical temperature turn out to be quite large, of the order of  $10 \sim 100$  MeV [7–21], compared to the core temperature of the neutron star, which is estimated to be  $\lesssim 0.7$  MeV [22]. Furthermore, it is found that, when the density is large enough for strange quark to participate in the Cooper-pairing, not only color symmetry but also chiral symmetry are spontaneously broken due to so-called color-flavor locking (CFL) [23]: At low temperature, the Cooper pairs of quarks form to lock the color and flavor indices as

$$\langle \psi_{L i \alpha}^a(\vec{p}) \psi_{L j \beta}^b(-\vec{p}) \rangle = - \langle \psi_{R i \alpha}^a(\vec{p}) \psi_{R j \beta}^b(-\vec{p}) \rangle = \epsilon_{\alpha \beta} \epsilon^{abI} \epsilon_{ijI} \Delta(p_F), \quad (1)$$

where  $a, b = 1, 2, 3$  and  $i, j = 1, 2, 3$  are color and flavor indices, respectively, and we ignore the small color sextet component in the condensate. In this CFL phase, the particle spectrum can be precisely mapped into that of the hadronic phase at low density. Observing this map, Schäfer and Wilczek [24,25] further conjectured that two phases are in fact continuously connected to each other. The CFL phase at high density is complementary to the hadronic phase at low density. This conjecture was subsequently supported [26] by showing that quarks in the CFL phase are realized as Skyrmions, called superqualitons, just like baryons are realized as Skyrmions in the hadronic phase.

Quark matter with a finite baryon number is described by QCD with a chemical potential, which is to restrict the system to have a fixed baryon number;

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} - \mu \bar{\psi}_i \gamma^0 \psi_i, \quad (2)$$

where  $\bar{\psi}_i \gamma^0 \psi_i$  is the quark number density and equal chemical potentials are assumed for different flavors, for simplicity. The ground state in the CFL phase is nothing but the Fermi sea where all quarks are gaped by the Cooper-pairing; the octet has a gap  $\Delta$  while the singlet has  $2\Delta$ . Equivalently, this system can be described in terms of bosonic degrees of freedom, which are small fluctuations of the Cooper pairs. Following the previous work [26], we introduce bosonic variables, defined as

$$U_{L ai}(x) \equiv \lim_{y \rightarrow x} \frac{|x - y|^{\gamma_m}}{\Delta(p_F)} \epsilon_{abc} \epsilon_{ijk} \psi_L^{bj}(-\vec{v}_F, x) \psi_L^{ck}(\vec{v}_F, y), \quad (3)$$

where  $\gamma_m$  ( $\sim \alpha_s$ ) is the anomalous dimension of the diquark field and  $\psi(\vec{v}_F, x)$  denotes a quark field with momentum close to a Fermi momentum  $\mu \vec{v}_F$  [14]. Similarly, we define  $U_R$  in terms of right-handed quarks to describe the small fluctuations of the condensate of right-handed quarks. Since the bosonic fields,  $U_{L,R}$ , are colored, they will interact with gluons. In fact, the colored massless excitations will constitute the longitudinal components of gluons through Higgs mechanism. Thus, the low-energy effective Lagrangian density for the bosonic fields in the CFL phase can be written as

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}^A F^{\mu\nu A} + g_s G_\mu^A J^{\mu A} + \left[ \frac{1}{4}F^2 \text{tr}(\partial_\mu U_L^\dagger \partial^\mu U_L) + n_L \mathcal{L}_{WZW} + (L \leftrightarrow R) \right] + \mathcal{L}_m + \dots, \quad (4)$$

where  $\mathcal{L}_m$  is the meson mass term and the ellipsis denotes the higher order terms in the derivative expansion, including mixing terms between  $U_L$  and  $U_R$ . The gluons couple to the bosonic fields through a minimal coupling with a conserved current, given as

$$J^{A\mu} = \frac{i}{2}F^2 \text{Tr} U_L^{-1} T^A \partial^\mu U_L + \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} T^A U_L^{-1} \partial_\nu U_L U_L^{-1} \partial_\rho U_L U_L^{-1} \partial_\sigma U_L + (L \leftrightarrow R) + \dots, \quad (5)$$

where the ellipsis denotes the currents from the higher order derivative terms in Eq. (4).  $F$  is a quantity analogous to the pion decay constant, calculated to be  $F \sim \mu$  in the CFL color superconductor [27]. The Wess-Zumino-Witten (WZW) term [28] is described by the action

$$\Gamma_{WZW} \equiv \int d^4x \mathcal{L}_{WZW} = -\frac{i}{240\pi^2} \int_M d^5r \epsilon^{\mu\nu\alpha\beta\gamma} \text{tr}(l_\mu l_\nu l_\alpha l_\beta l_\gamma) \quad (6)$$

where  $l_\mu = U_L^\dagger \partial_\mu U_L$  and the integration is defined on a five-dimensional manifold  $M = V \otimes S^1 \otimes I$  with the three dimensional space  $V$ , the compactified time  $S^1$ , and a unit interval  $I$  needed for the local form of WZW term. The coefficients of the WZW terms in the effective Lagrangian (4) have been shown to be  $n_{L,R} = 1$  by matching the flavor anomalies [26], which is later confirmed by an explicit calculation [29].

Among the small fluctuations of condensates, the colorless excitations correspond to genuine Nambu-Goldstone (NG) bosons, which can be described by a color singlet combination of  $U_{L,R}$  [30,31], given as

$$\Sigma_i^j \equiv U_{Lai} U_R^{*aj}. \quad (7)$$

The NG bosons transform under the  $SU(3)_L \times SU(3)_R$  chiral symmetry as

$$\Sigma \mapsto g_L \Sigma g_R^\dagger, \quad \text{with } g_{L,R} \in SU(3)_{L,R}. \quad (8)$$

Since the chiral symmetry is explicitly broken by the current quark mass, the instanton effects, and the electromagnetic interaction, the NG bosons will get mass, which has been calculated by various groups [27,30,32,33]. Here we focus on the meson mass due to the current strange quark mass ( $m_s$ ), since it will be dominant for the intermediate density. Then, the meson mass term is simplified as

$$\mathcal{L}_m = C \text{tr}(M^T \Sigma) \cdot \text{tr}(M^* \Sigma^\dagger) + O(M^4), \quad (9)$$

where  $M = \text{diag}(0, 0, m_s)$  and  $C \sim \Delta^4/\mu^2 \cdot \ln(\mu^2/\Delta^2)$ . (Note that in general there will be two more mass terms quadratic in  $M$ . But, they all vanish if we neglect the current mass of up and down quarks and also the small color-sextet component of the Cooper pair [30].)

Now, let us try to describe the CFL color superconductor in terms of the bosonic variables. We start with the effective Lagrangian (4), which is good at low energy, without putting in the quark fields. As in the Skyrme model of baryons, we anticipate the gaped quarks come out as solitons, made of the bosonic degrees of freedom. That the Skyrme picture can be realized in the CFL color superconductor is already shown in [26], but there

the mass of the soliton is not properly calculated. Here, by identifying the correct ground state of the CFL superconductor in the bosonic description, we find the superqualitons have same quantum numbers as quarks with mass of the order of gap, showing that they are really the gaped quarks in the CFL color superconductor. Furthermore, upon quantizing the zero modes of the soliton, we find that high spin excitations of the soliton have energy of order of  $\mu$ , way beyond the scale where the effective bosonic description is applicable, which we interpret as the absence of high-spin quarks, in agreement with the fermionic description. It is interesting to note that, as we will see below, by calculating the soliton mass in the bosonic description, one finds the coupling and the chemical potential dependence of the Cooper-pair gap, at least numerically, which gives us a complementary way, if not better, of estimating the gap.

As the baryon number (or the quark number) is conserved, though spontaneously broken,<sup>1</sup> the ground state in the bosonic description should have the same baryon (or quark) number as the ground state in the fermionic description. Under the  $U(1)_Q$  quark number symmetry, the bosonic fields transform as

$$U_{L,R} \mapsto e^{i\theta Q} U_{L,R} e^{-i\theta Q} = e^{2i\theta} U_{L,R}, \quad (10)$$

where  $Q$  is the quark number operator, given in the bosonic description as

$$Q = i \int d^3x \frac{F^2}{4} \text{Tr} \left[ U_L^\dagger \partial_t U_L - \partial_t U_L^\dagger U_L + (L \leftrightarrow R) \right], \quad (11)$$

neglecting the quark number coming from the WZW term, since the ground state has no nontrivial topology. The energy in the bosonic description is

$$E = \int d^3x \frac{F^2}{4} \text{Tr} \left[ |\partial_t U_L|^2 + |\vec{\nabla} U_L|^2 + (L \leftrightarrow R) \right] + E_m + \delta E, \quad (12)$$

where  $E_m$  is the energy due to meson mass and  $\delta E$  is the energy coming from the higher derivative terms. Assuming the meson mass energy is positive and  $E_m + \delta E \geq 0$ , which is reasonable because  $\Delta/F \ll 1$ , we can take, dropping the positive terms due to the spatial derivative,

$$E \geq \int d^3x \frac{F^2}{4} \text{Tr} \left[ |\partial_t U_L|^2 + (L \leftrightarrow R) \right] (\equiv E_Q). \quad (13)$$

Since for any number  $\alpha$

$$\int d^3x \text{Tr} \left[ |U_L + \alpha i \partial_t U_L|^2 + (L \leftrightarrow R) \right] \geq 0, \quad (14)$$

we get a following Schwartz inequality,

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<sup>1</sup>The spontaneously broken baryon number just means that the states in the Fock space do not have a well-defined baryon number. But, still the baryon number current is conserved in the operator sense [34].

$$Q^2 \leq I E_Q, \quad (15)$$

where we defined

$$I = \frac{F^2}{4} \int d^3x \operatorname{Tr} [U_L U_L^\dagger + (L \leftrightarrow R)]. \quad (16)$$

Note that the lower bound in Eq. (15) is saturated for  $E_Q = \omega Q$  or

$$U_{L,R} = e^{i\omega t} \quad \text{with} \quad \omega = \frac{Q}{I}. \quad (17)$$

The ground state of the color superconductor, which has the lowest energy for a given quark number  $Q$ , is nothing but a so-called  $Q$ -matter, or the interior of a very large  $Q$ -ball [35,36]. Since in the fermionic description the system has the quark number  $Q = \mu^3/\pi^2 \int d^3x = \mu^3/\pi^2 \cdot I/F^2$ , we find, using  $F \simeq 0.209\mu$  [27],

$$\omega = \frac{1}{\pi^2} \left( \frac{\mu}{F} \right)^3 F \simeq 2.32\mu. \quad (18)$$

By passing, we note that  $\omega$  is numerically very close to  $4\pi F$ . The ground state of the system in the bosonic description is a  $Q$ -matter whose energy per unit quark number is  $\omega$ . Now, let us suppose we consider creating a  $Q = 1$  state out of the ground state. In the fermionic description, this corresponds that we excite a gaped quark in the Fermi sea into a free state, which costs energy at least  $2\Delta$ . In the bosonic description, this amounts to creating a superqualiton out of the  $Q$ -matter, while reducing the quark number of the  $Q$ -matter by one. Therefore, since, reducing the quark number of the  $Q$ -matter by one, we gain energy  $\omega$ , the energy cost to create a gaped quark from the ground state in the bosonic description is

$$\delta\mathcal{E} = M_Q - \omega, \quad (19)$$

where  $M_Q$  is the energy of the superqualiton configuration. From the relation that  $2\Delta = M_Q - \omega$ , later we estimate numerically the coupling and the chemical potential dependence of the Cooper gap.

Following the Skyrme picture of baryons in QCD at low density, we now investigate how gaped quarks in high density QCD are realized in its bosonic description with the Lagrangian given in Eq. (4) [26]. Assuming the maximal symmetry in the superqualiton, we seek a static configuration for the field  $U_L$  which is the  $SU(2)$  hedgehog in color-flavor in  $SU(3)$

$$U_{Lc}(\vec{x}) = \begin{pmatrix} e^{i\vec{\tau} \cdot \hat{x} \theta(r)} & 0 \\ 0 & 1 \end{pmatrix} \quad (20)$$

where the  $\tau_i$  ( $i=1,2,3$ ) are Pauli matrices,  $\hat{x} \equiv \vec{x}/r$  and  $\theta(r)$  is the chiral angle determined by minimizing the static mass  $M_0$  given below and for unit winding number we take  $\theta(r = \infty) = 0$  and  $\theta(0) = \pi$ . The static configuration for the other fields are described as

$$U_R = 0, \quad G_0^A = \frac{x^A}{r} \omega(r), \quad G_i^A = 0. \quad (21)$$

Now we consider the zero modes of the  $SU(3)$  superqualiton as follows

$$U(\vec{x}, t) = \mathcal{A}(t)U_{Lc}(\vec{x})\mathcal{A}(t)^\dagger. \quad (22)$$

The Lagrangian for the zero modes is then given by

$$L = -M_0 + \frac{1}{2}I_{ab}\text{tr}(\mathcal{A}^\dagger \dot{\mathcal{A}} \frac{\lambda_a}{2})\text{tr}(\mathcal{A}^\dagger \dot{\mathcal{A}} \frac{\lambda_b}{2}) - \frac{i}{2}\text{tr}(Y\mathcal{A}^\dagger \dot{\mathcal{A}}), \quad (23)$$

where  $I_{ab}$  is an invariant tensor on  $\mathcal{M} = SU(3)/U(1)$  and  $Y$  is the hypercharge

$$Y = \frac{\lambda_8}{\sqrt{3}} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Using the above static configuration, we obtain the static mass  $M_0$  and the tensor  $I_{ab}$  as follows

$$\begin{aligned} M_0 &= \frac{4\pi}{3}F^2 \int_0^\infty dr \left[ \frac{1}{2}r^2 \left( \frac{d\theta}{dr} \right)^2 + \sin^2 \theta + \frac{\alpha_s}{2\pi^3 F^2} \left( \frac{\theta - \sin \theta \cos \theta - \pi}{2r} \right)^2 e^{-2m_E r} \right] \\ I_{ab} &= -\frac{32\pi}{9}F^2 \int_0^\infty dr r^2 \sin^2 \theta = -4I_1 \quad (a = b = 1, 2, 3) \\ &= -\frac{8\pi}{3}F^2 \int_0^\infty dr r^2 (1 - \cos \theta) = -4I_2 \quad (a = b = 4, 5, 6, 7) \\ &= 0 \quad (a = b = 8) \end{aligned} \quad (24)$$

where  $\alpha_s$  is the strong coupling constant and  $m_E = \mu(6\alpha_s/\pi)^{1/2}$  is the electric screening mass for the gluons.

Since  $\mathcal{A}$  belongs to  $SU(3)$ ,  $\mathcal{A}^\dagger \dot{\mathcal{A}}$  is anti-Hermitian and traceless to be expressed as a linear combination of  $i\lambda_a$  as follows

$$\mathcal{A}^\dagger \dot{\mathcal{A}} = iFv^a \lambda_a = iF \begin{pmatrix} \vec{v} \cdot \tau + \nu 1 & V \\ V^\dagger & -2\nu \end{pmatrix}$$

where

$$\vec{v} = (v^1, v^2, v^3), \quad V = \begin{pmatrix} v^4 - iv^5 \\ v^6 - iv^7 \end{pmatrix}, \quad \nu = \frac{v^8}{\sqrt{3}}. \quad (25)$$

The Lagrangian is then expressed as

$$L = -M_0 + 2F^2 I_1 \vec{v}^2 + 2F^2 I_2 V^\dagger V + \frac{1}{3}NF\nu. \quad (26)$$

In order to separate the  $SU(2)$  rotations from the deviations into strange directions, we write the time-dependent rotations as follows

$$\mathcal{A}(t) = \begin{pmatrix} A(t) & 0 \\ 0 & 1 \end{pmatrix} S(t)$$

with  $A(t) \in \text{SU}(2)$  and the small rigid oscillations  $S(t)$  around the  $\text{SU}(2)$  rotations. Furthermore, in the  $\text{SU}(2)$  subgroup of  $\text{SU}(3)$ , we exploit the time-dependent collective coordinates  $a^\mu = (a^0, \vec{a})$  ( $\mu = 0, 1, 2, 3$ ) as in the  $\text{SU}(2)$  Skyrmion [37]

$$A(t) = a^0 + i\vec{a} \cdot \vec{\tau}.$$

On the other hand the small rigid oscillations  $S$ , which were also used in Ref. [38], can be described as

$$S(t) = \exp(i \sum_{a=4}^7 d^a \lambda_a) = \exp(i\mathcal{D}),$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & \sqrt{2}D \\ \sqrt{2}D^\dagger & 0 \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} d^4 - id^5 \\ d^6 - id^7 \end{pmatrix}.$$

After some algebra, one can obtain the relations among the variables in (25) and the  $\text{SU}(2)$  collective coordinates  $a^\mu$  and the strange deviations  $D$  such as

$$\begin{aligned} F\nu &= \frac{i}{2}(D^\dagger \dot{D} - \dot{D}^\dagger D) - D^\dagger(a^0 \vec{a} - \dot{a}^0 \vec{a} + \vec{a} \times \dot{\vec{a}}) \cdot \vec{\tau} D \\ &\quad - \frac{i}{3}(D^\dagger \dot{D} - \dot{D}^\dagger D)D^\dagger D + \dots, \end{aligned} \quad (27)$$

to yield the superqualiton Lagrangian to order  $1/N$

$$\begin{aligned} L &= -M_0 + 2I_1 \dot{a}^\mu \dot{a}^\mu + 4I_2 \dot{D}^\dagger \dot{D} + \frac{i}{6}N(D^\dagger \dot{D} - \dot{D}^\dagger D) - 4I_2 m_K^2 D^\dagger D \\ &\quad + 2i(I_1 - 2I_2)\{D^\dagger(a^0 \vec{a} - \dot{a}^0 \vec{a} + \vec{a} \times \dot{\vec{a}}) \cdot \vec{\tau} \dot{D} \\ &\quad - \dot{D}^\dagger(a^0 \vec{a} - \dot{a}^0 \vec{a} + \vec{a} \times \dot{\vec{a}}) \cdot \vec{\tau} D\} - \frac{1}{3}ND^\dagger(a^0 \vec{a} - \dot{a}^0 \vec{a} + \vec{a} \times \dot{\vec{a}}) \cdot \vec{\tau} D \\ &\quad + 2\left(I_1 - \frac{4}{3}I_2\right)(D^\dagger D)(\dot{D}^\dagger \dot{D}) - \frac{1}{2}\left(I_1 - \frac{4}{3}I_2\right)(D^\dagger \dot{D} + \dot{D}^\dagger D)^2 \\ &\quad + 2I_2(D^\dagger \dot{D} - \dot{D}^\dagger D)^2 - \frac{i}{9}N(D^\dagger \dot{D} - \dot{D}^\dagger D)D^\dagger D \\ &\quad + \frac{8}{3}I_2 m_K^2 (D^\dagger D)^2 \end{aligned} \quad (28)$$

where we have included the kaon mass terms proportional to the strange quark mass which is not negligible.

The momenta  $\pi^\mu$  and  $\pi_s^\alpha$ , conjugate to the collective coordinates  $a^\mu$  and the strange deviation  $D_\alpha^\dagger$  are given by

$$\begin{aligned} \pi^0 &= 4I_1 \dot{a}^0 - 2i(I_1 - 2I_2)(D^\dagger \vec{a} \cdot \vec{\tau} \dot{D} - \dot{D}^\dagger \vec{a} \cdot \vec{\tau} D) + \frac{1}{3}ND^\dagger \vec{a} \cdot \vec{\tau} D \\ \vec{\pi} &= 4I_1 \dot{\vec{a}} + 2i(I_1 - 2I_2)\{D^\dagger(a^0 \vec{\tau} - \vec{a} \times \vec{\tau})\dot{D} - \dot{D}^\dagger(a^0 \vec{\tau} - \vec{a} \times \vec{\tau})D\} \\ &\quad - \frac{1}{3}ND^\dagger(a^0 \vec{\tau} - \vec{a} \times \vec{\tau})D \end{aligned}$$

$$\begin{aligned}
\pi_s = & 4I_2\dot{D} - \frac{i}{6}ND - 2i(I_1 - 2I_2)(a^0\vec{a} - \dot{a}^0\vec{a} + \vec{a} \times \vec{\dot{a}}) \cdot \vec{\tau}D \\
& + 2\left(I_1 - \frac{4}{3}I_2\right)(D^\dagger D)\dot{D} - \left(I_1 - \frac{4}{3}I_2\right)(D^\dagger\dot{D} + \dot{D}^\dagger D)D \\
& - 4I_2(D^\dagger\dot{D} - \dot{D}^\dagger D)D + \frac{i}{9}N(D^\dagger D)D
\end{aligned}$$

which satisfy the Poisson brackets

$$\{a^\mu, \pi^\nu\} = \delta^{\mu\nu}, \quad \{D_\alpha^\dagger, \pi_s^\beta\} = \{D^\beta, \pi_{s,\alpha}^\dagger\} = \delta_\alpha^\beta.$$

Performing Legendre transformation, we obtain the Hamiltonian to order  $1/N$  as follows

$$\begin{aligned}
H = & M_0 + \frac{1}{8I_1}\pi^\mu\pi^\mu + \frac{1}{4I_2}\pi_s^\dagger\pi_s - i\frac{N}{24I_2}(D^\dagger\pi_s - \pi_s^\dagger D) + \left(\frac{N^2}{144I_2}\right. \\
& + 4I_2m_K^2) D^\dagger D + i\left(\frac{1}{4I_1} - \frac{1}{8I_2}\right)\{D^\dagger(a^0\vec{\pi} - \vec{a}\pi^0 + \vec{a} \times \vec{\pi}) \cdot \vec{\tau}\pi_s \\
& - \pi_s^\dagger(a^0\vec{\pi} - \vec{a}\pi^0 + \vec{a} \times \vec{\pi}) \cdot \vec{\tau}D\} + \frac{N}{24I_2}D^\dagger(a^0\vec{\pi} - \vec{a}\pi^0 + \vec{a} \times \vec{\pi}) \cdot \vec{\tau}D \\
& + \left(\frac{1}{2I_1} - \frac{1}{3I_2}\right)(D^\dagger D)(\pi_s^\dagger\pi_s) + \left(\frac{1}{12I_2} - \frac{1}{8I_1}\right)(D^\dagger\pi_s + \pi_s^\dagger D)^2 \\
& - \frac{1}{8I_2}(D^\dagger\pi_s - \pi_s^\dagger D)^2 - i\frac{N}{24I_2}(D^\dagger\pi_s - \pi_s^\dagger D)(D^\dagger D) \\
& + \left(\frac{N^2}{108I_2} - \frac{8}{3}I_2m_K^2\right)(D^\dagger D)^2.
\end{aligned} \tag{29}$$

Applying the Batalin-Fradkin-Tyutin (BFT) scheme [39,40] to the above result, one can obtain the first class Hamiltonian

$$\begin{aligned}
\tilde{H} = & M_0 + \frac{1}{8I_1}(\pi^\mu - a^\mu\Phi^2)(\pi^\mu - a^\mu\Phi^2)\frac{a^\nu a^\nu}{a^\nu a^\nu + 2\Phi^1} \\
& + \frac{1}{4I_2}\pi_s^\dagger\pi_s - i\frac{N}{24I_2}(D^\dagger\pi_s - \pi_s^\dagger D) + \left(\frac{N^2}{144I_2} + 4I_2m_K^2\right) D^\dagger D \\
& + i\left(\frac{1}{4I_1} - \frac{1}{8I_2}\right)\{D^\dagger(a^0\vec{\pi} - \vec{a}\pi^0 + \vec{a} \times \vec{\pi}) \cdot \vec{\tau}\pi_s \\
& - \pi_s^\dagger(a^0\vec{\pi} - \vec{a}\pi^0 + \vec{a} \times \vec{\pi}) \cdot \vec{\tau}D\} + \frac{N}{24I_2}D^\dagger(a^0\vec{\pi} - \vec{a}\pi^0 + \vec{a} \times \vec{\pi}) \cdot \vec{\tau}D \\
& + \dots
\end{aligned} \tag{30}$$

where the ellipsis stands for the strange-strange interaction terms of order  $1/N$  which can be readily read off from Eq. (29).

Following the Klebanov and Westerberg's quantization scheme [38] for the strangeness flavor direction one can obtain the Hamiltonian of the form

$$\tilde{H} = M_0 + \nu a^\dagger a + \frac{1}{2I_1}\left(\vec{I}^2 + 2c\vec{I} \cdot \vec{J}_s + \bar{c}\vec{J}_s^2 + \frac{1}{4}\right) \tag{31}$$

where  $\vec{I}$  and  $\vec{J}_s$  are the isospin and angular momentum for the strange quarks and



$$\begin{aligned}
\nu &= \frac{N}{24I_2}(\mu_K - 1) \\
c &= 1 - \frac{I_1}{2I_2\mu_K}(\mu_K - 1) \\
\bar{c} &= 1 - \frac{I_1}{I_2\mu_K^2}(\mu_K - 1)
\end{aligned}$$

with

$$\mu_K = \left(1 + \frac{m_K^2}{m_0^2}\right)^{1/2}, \quad m_0 = \frac{N}{24I_2}.$$

Here note that  $a^\dagger$  is creation operator for constituent strange quarks and the factor  $\frac{1}{4}$  originates from the BFT corrections [40], which are applicable to only u- and d-superqualitons. The Hamiltonian (31) then yields the mass spectrum of superqualiton as follows

$$\begin{aligned}
M_Q = M_0 - (Y - \frac{1}{3})\nu + \frac{1}{2I_1} [cJ(J+1) + (1-c)I(I+1) \\
+ (\bar{c} - c)\frac{(Y - 1/3)(Y - 7/3)}{4} + \frac{1}{4}\delta_{I,1/2}]
\end{aligned} \tag{32}$$

with the total angular momentum of the quark  $\vec{J} = \vec{I} + \vec{J}_s$ .

Unlike creating the Skyrmions out of the Dirac vacuum, in dense matter the energy cost to create a superqualiton should be compared with the Fermi Sea. By creating a superqualiton, we have to remove one quark in the Fermi sea since the total Baryon number has to remain unchanged. Similar to the Cooper pair mechanism [41], from Eq. (19), the twice of u- and s-superqualiton masses are then given by

$$\begin{aligned}
2M_u &= M_0 + \frac{1}{2I_1} - \omega \\
2M_s &= M_0 + \nu + \frac{3}{8I_1}\bar{c} - \omega
\end{aligned} \tag{33}$$

to yield the predictions for the values of  $M_u(=M_d)$  and  $M_s$

$$\begin{aligned}
M_u &= 0.079 \times 4\pi F, & M_s &= 0.081 \times 4\pi F, & \text{for } m_K/F &= 0.1 \\
M_u &= 0.079 \times 4\pi F, & M_s &= 0.089 \times 4\pi F, & \text{for } m_K/F &= 0.3 \\
M_u &= 0.079 \times 4\pi F, & M_s &= 0.109 \times 4\pi F, & \text{for } m_K/F &= 0.8.
\end{aligned} \tag{34}$$

To see if the estimated superqualiton mass is indeed the Cooper gap, one needs to compare our numerical results with the analytic expression for the coupling dependence of the gap. In Table 1 we show the dependence of superqualiton masses on the strong coupling constant  $\alpha_s$ . By fitting the numerical results with the gap as, in the unit of  $4\pi F$ ,

$$\log(M_u) = a \log(\alpha_s) + b\alpha_s^{-1/2} + c, \tag{35}$$

we get  $a = 0.0135$ ,  $b = 0.00341$ , and  $c = -2.53$ . This is very different from the analytic expression obtained in the weak coupling limit [13–18],

$$\Delta \sim \frac{\mu}{g_s^5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g_s}\right). \quad (36)$$

As suggested in the reference [42], the weak coupling result (36) may be applicable only when the coupling is extremely small or the chemical potential is very large. In our numerical analysis, we are unable to probe this region.

In conclusion, we have bosonized the CFL phase of QCD at high density, where elementary excitations are pions and kaons. The ground state is shown to be a  $Q$ -matter, whose energy per unit quark number is  $2.32\mu$ . The gaped quarks are realized as solitons, so-called superqualitons. The energy to create a superqualiton out of the ground state is argued to be twice of the gap, which is checked numerically by calculating the superqualiton mass as the coupling changes. Finally, we have quantized the zero modes of superqualiton and find that the mass of high-spin states is larger than the chemical potential, which is interpreted as an absence of such states in the bosonized theory, in agreement with the fermionic description.

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# TABLES

TABLE I. The dependence of qualiton masses on the coupling  $\alpha_s$  with  $m_K/F = 0.3$

$\alpha_s$	$M_Q(u)/4\pi F$	$M_Q(s)/4\pi F$	$M_u/4\pi F$	$M_s/4\pi F$
0.050	1.040	1.061	0.078	0.089
0.100	1.040	1.061	0.078	0.089
0.150	1.041	1.061	0.079	0.089
0.200	1.041	1.061	0.079	0.089
0.250	1.041	1.061	0.079	0.089
0.300	1.041	1.062	0.079	0.089
0.350	1.041	1.062	0.079	0.089
0.400	1.042	1.062	0.079	0.089
0.450	1.042	1.062	0.079	0.089
0.500	1.042	1.062	0.079	0.089
0.550	1.042	1.062	0.079	0.089
0.600	1.042	1.062	0.079	0.089
0.650	1.042	1.062	0.079	0.090
0.700	1.042	1.063	0.079	0.090
0.750	1.042	1.062	0.079	0.090
0.800	1.042	1.063	0.079	0.090
0.850	1.042	1.063	0.079	0.090
0.900	1.042	1.063	0.079	0.090
0.950	1.042	1.063	0.080	0.090
1.000	1.043	1.063	0.080	0.090